EASY PROOF OF THE JACOBIAN FOR THE N-DIMENSIONAL POLAR COORDINATES

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Abstract. The n-dimensional polar coordinates are defined and a short elegant proof of their Jacobian is given. A method, which we call perturbative method, is used to preserve the elegant approach in all possible special cases.

1. Introduction.

In this note, a very short and elementary proof of the Jacobian for the n-dimensional polar coordinates is given. Various proofs for that Jacobian exist in the literature, but they are very big, technical, and they are dependent on voluminous preliminary theories. Some authors erroneously assume that it is possible to find a very easy proof by using mathematical induction without doing it. In fact, mathematical induction cannot be used for there is a lack of a recursive relation between the Jacobians in the general case.

In the 1960's, Vilenkin, Kuznetsov, and Smorodinskii [7] talk about "polyspherical coordinates", but do not provide the proof for the Jacobian. Later on, S. Hassani [2] claims that the proof of the Jacobian could be done by mathematical induction, but does not provide a proof. C. Cumbus et. al. [1], L. E. Espinola López et. al. [4], and B. K. Novosadov et. al. [5] simply state the Jacobian. W. D. Richter [6] introduces "generalized spherical and simplicial coordinates" and provides the proof of the Jacobian for these coordinates. His proof comes as a result of a very technical and voluminous (13 pages) theory. A. P. Lehnen, on his website [3], in the appendix, does give the proof of the Jacobian for the "n-dimensional spherical coordinates." Not only that his proof comes as a result of a long and technical (15 pages) theory, but also, as expected, it is not done by mathematical induction.

The type of coordinates chosen in this paper is the same as the one in [3]. It belongs to a family of similar coordinates, used to describe points in the n-dimensional space, which was introduced by Vilenkin, Kuznetsov and Smorodinskii [7]. If desired, this type of coordinates could very easily be converted to resemble the "standard" n-dimensional polar coordinates. One gets the standard polar and spherical coordinates, as special cases, for $n = 2$ and 3 respectively, by a simple substitution of the first polar angle $\theta = \frac{\pi}{2} - \theta_1$ and keeping the rest of the coordinates the same. This substitution would result in the Jacobian being multiplied by -1.

In Sections 2, the n-dimensional polar coordinates are introduced. In Section 3, the proof of the Jacobian is presented. It is short and elementary, but initially it could be accomplished only in the case when $\tan\theta_k$ and $\cot\theta_k$ are all defined (and different than 0) for all θ_k $(k = 1, 2, 3, ..., n-1.)$ This obstacle is overcome by a special manipulation that we call a perturbative (not perturbation) method. This

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perturbative method makes it possible to preserve and extend the initial proof to all possible cases and it is also illustrated in Section 3.

2. Generalization to the n-dimensional case, the n-d polar coordinates.

Definition. For $n \geq 2$, and for any point $P(x_1, x_2, ..., x_n)$ in the $x_1 - x_2 - ... - x_n$ space, $[r, \theta_1, \theta_2, ..., \theta_{n-1}]$, where $r \geq 0$, are referred to as the n-d polar coordinates of point $P(x_1, x_2, ..., x_n)$, if

$$\begin{cases} x_1 & = r \sin\theta_1 \sin\theta_2 ... \sin\theta_{n-1} \\ x_2 & = r \cos\theta_1 \sin\theta_2 ... \sin\theta_{n-1} \\ x_3 & = r \cos\theta_2 \sin\theta_3 ... \sin\theta_{n-1} \\ ... & \\ x_{n-1} & = r \cos\theta_{n-2} \sin\theta_{n-1} \\ x_n & = r \cos\theta_{n-1} \end{cases} \quad (0 \leq \theta_1 < 2\pi \ , \ 0 \leq \theta_j \leq \pi, \ j = 2, 3, ...n-1) \tag{1}$$

Remark. Evidently, $r = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. This shows that r is the Euclidean distance from $P(x_1, x_2, ..., x_n)$ to $O(0, 0, ..., 0)$.

In general, the Jacobian of the n-d polar coordinates is given by

$$J_n = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(r, \theta_1, \theta_2, ..., \theta_{n-1})} = \det[A_{n,1}, B_{n,n-1}] \tag{2}$$

where $A_{n,1} = \begin{bmatrix} \frac{\partial x_1}{\partial r} \\ \frac{\partial x_2}{\partial r} \\ \vdots \\ \frac{\partial x_n}{\partial r} \end{bmatrix}$ and $B_{n,n-1} = \begin{bmatrix} \frac{\partial x_m}{\partial \theta_k} \end{bmatrix}$ for $m = 1, ..., n$ and $k = 2, ..., n-1$.

3. Evaluation of the Jacobian for the n-dimensional polar coordinates.

For $n = 2$,

$$J_2 = \frac{\partial(x_1, x_2)}{\partial(r, \theta_1)} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta_1} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta_1} \end{vmatrix} = \begin{vmatrix} \sin\theta_1 & r\cos\theta_1 \\ \cos\theta_1 & -r\sin\theta_1 \end{vmatrix} = -r \tag{3}$$

For $n \geq 3$, (1) could be written as

$$\begin{cases} x_1 & = r \prod_{k=1}^{n-1} \sin\theta_k \\ x_m & = r \cos\theta_{m-1} \prod_{k=m}^{n-1} \sin\theta_k, \quad 2 \leq m \leq n-1 \\ x_n & = r \cos\theta_{n-1} \end{cases} \tag{4}$$

For $r = 0$, $J_n = 0$.

For $r > 0$, our approach for evaluating the Jacobian of the n-dimensional polar coordinates uses some manipulations that could be accomplished only in the case that $\tan\theta_k$ and $\cot\theta_k$ are all defined (and different than 0) for all θ_k $(k = 1, 2, 3, ..., n-1.)$ In order to facilitate the proof and preserve this approach in all possible cases, a special manipulation, which we call a perturbative method, is used. This method is shown below.

Let

$$0 < \epsilon < \frac{1}{2}\min_k\{d_k\} \tag{5}$$

where d_k $(k = 1, 2, 3, ..., n-1)$ is the shortest distance from θ_k to the numbers of the form $\frac{l\pi}{2}$ $(l \in \mathbb{Z})$ which are different than θ_k and

$$\theta_k^\star = \theta_k^\star(\epsilon) = \theta_k + \epsilon \tag{6}$$

Thus,

$$\theta_k^\star \neq \frac{l\pi}{2}\ (\forall l \in \mathbb{Z}) \tag{7}$$

and $\tan\theta_k^\star$ and $\cot\theta_k^\star$ are all defined (and different than zero.)

Replacing θ_k by $\theta_k^\star$ $(k = 1, 2, 3, ..., n-1)$, (4) becomes

$$\begin{cases} x_1^\star & = r\prod_{k=1}^{n-1}\sin\theta_k^\star \\ x_m^\star & = r\cos\theta_{m-1}^\star\prod_{k=m}^{n-1}\sin\theta_k^\star, \quad 2 \leq m \leq n-1 \\ x_n^\star & = r\cos\theta_{n-1}^\star \end{cases} \tag{8}$$

Now, for $r > 0$, one obtains the following:

$$\frac{\partial x_m^\star}{\partial r} = \frac{1}{r}x_m^\star, \quad 1 \leq m \leq n \tag{9}$$

$$\frac{\partial x_m^\star}{\partial \theta_k^\star} = x_m^\star\cot\theta_k^\star, \quad 1 \leq m \leq n,\ m \leq k \leq n-1 \tag{10}$$

$$\frac{\partial x_m^\star}{\partial \theta_k^\star} = 0, \quad 3 \leq m \leq n,\ 1 \leq k \leq m-2 \tag{11}$$

$$\frac{\partial x_m^\star}{\partial \theta_{m-1}^\star} = -x_m\tan\theta_{m-1}^\star, \quad 2 \leq m \leq n \tag{12}$$

Therefore,

$$\begin{aligned} J_n^\star &= \frac{\partial(x_1^\star, x_2^\star, ..., x_n^\star)}{\partial(r, \theta_1^\star, \theta_2^\star, ..., \theta_{n-1}^\star)} \\ &= \begin{vmatrix} \frac{x_1^\star}{r} & x_1^\star\cot\theta_1^\star & x_1^\star\cot\theta_2^\star & ... & x_1^\star\cot\theta_{n-2}^\star & x_1^\star\cot\theta_{n-1}^\star \\ \frac{x_2^\star}{r} & -x_2^\star\tan\theta_1^\star & x_2^\star\cot\theta_2^\star & ... & x_2^\star\cot\theta_{n-2}^\star & x_2^\star\cot\theta_{n-1}^\star \\ \frac{x_3^\star}{r} & 0 & -x_3^\star\tan\theta_2^\star & ... & x_3^\star\cot\theta_{n-2}^\star & x_3^\star\cot\theta_{n-1}^\star \\ ... & ... & ... & ... & ... & ... \\ \frac{x_{n-1}^\star}{r} & 0 & 0 & ... & -x_{n-1}^\star\tan\theta_{n-2}^\star & x_{n-1}^\star\cot\theta_{n-1}^\star \\ \frac{x_n^\star}{r} & 0 & 0 & ... & 0 & -x_n^\star\tan\theta_{n-1}^\star \end{vmatrix} \end{aligned} \tag{13}$$

Pulling out $\frac{1}{r}$ from the 1^{st} column, $\cot\theta_k^\star$ from the $k{+}1^{st}$column ($k=1,2,3,...,n-1$), and $x_k^\star$ from the k^{th} row ($k=1,2,3,...,n$), (13) becomes

$$J_n^\star = \frac{1}{r}x_1^\star x_2^\star ... x_n^\star \cot\theta_1^\star \cot\theta_2^\star ... \cot\theta_{n-1}^\star \Delta_n^\star \tag{14}$$

where

$$\Delta_n^\star = \begin{vmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & -\tan^2\theta_1^\star & 1 & ... & 1 & 1 \\ 1 & 0 & -\tan^2\theta_2^\star & ... & 1 & 1 \\ ... & ... & ... & ... & ... & ... \\ 1 & 0 & 0 & ... & -\tan^2\theta_{n-2}^\star & 1 \\ 1 & 0 & 0 & ... & 0 & -\tan^2\theta_{n-1}^\star \end{vmatrix} \tag{15}$$

Subtracting the 1^{st} column of $\Delta_n^\star$ from the rest of the columns, yields

$$\Delta_n^\star = \begin{vmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 1 & -\sec^2\theta_1^\star & 0 & ... & 0 & 0 \\ 1 & -1 & -\sec^2\theta_2^\star & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 1 & -1 & -1 & ... & -\sec^2\theta_{n-2}^\star & 0 \\ 1 & -1 & -1 & ... & -1 & -\sec^2\theta_{n-1}^\star \end{vmatrix} \tag{16}$$

which is a lower triangular determinant.

Therefore,

$$\Delta_n^\star = (-1)^{n-1}\prod_{k=1}^{n-1}\sec^2\theta_k^\star \tag{17}$$

Thus,

$$J_n^\star = \frac{1}{r}r^n \prod_{k=1}^{n-1}\cos\theta_k^\star \prod_{k=1}^{n-1}\sin^k\theta_k^\star (-1)^{n-1}\prod_{k=1}^{n-1}\frac{1}{\cos^2\theta_k^\star}\prod_{k=1}^{n-1}\frac{\cos\theta_k^\star}{\sin\theta_k^\star}$$

which simplifies to

$$J_n^\star = (-1)^{n-1}r^{n-1}\prod_{k=2}^{n-1}\sin^{k-1}\theta_k^\star \tag{18}$$

Since $J_n^\star$ is a continous function of ϵ,

$$J_n = \lim_{\epsilon\to+0} J_n^\star = \lim_{\epsilon\to+0}(-1)^{n-1}r^{n-1}\prod_{k=2}^{n-1}\sin^{k-1}\theta_k^\star = \lim_{\epsilon\to+0}(-1)^{n-1}r^{n-1}\prod_{k=2}^{n-1}\sin^{k-1}(\theta_k+\epsilon) \tag{19}$$

Thus,

$$J_n = (-1)^{n-1}r^{n-1}\prod_{k=2}^{n-1}\sin^{k-1}\theta_k \tag{20}$$

This result remains valid also when $r=0$, since $J_n = 0$ in that case. This concludes the evaluation/proof of the Jacobian J_n.

4. Conclusions.

In this note, a short elegant proof of the Jacobian for the n-dimensional polar coordinates is presented. Our direct approach for the evaluation of the Jacobian does not require any advanced technical theorem/knowledge. A perturbative method is used to generalize this approach to all possible special cases.

References

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